Separability and Functional Structure of the Dual Profit Function in the Binary Partition

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ABSTRACT: *In this article it is shown that in the common case of a binary partition of the input or output variable vector (with the respective other vector remaining unpartitioned) strong separability in conjunction with strong separability in the extended partition is equivalent to a CES structure of the dual profit function – in opposition to any other economic behavioral function: the proof of the presented theorem about the profit function cannot be transferred to the dual cost or revenue function or any primal function.*

1 Introduction

Although it may appear to be only a minor special case, the partition of the input quantity or price vector into two rather than three or more sectors in order to postulate their separability among each other and of the output variable, i. e. separable inputs in the binary and extended partition, is of extraordinary importance in applied studies: any demand model concentrating on some interdependent inputs and neglecting the rest implicitly assumes a highly separable structure of the industry's overall behavioral function, namely an additive form.[*] Even more, general additivity is not enough: the basic function must be literally additive, i. e. the plain sum of one function of the input variables of interest alone and another function of all other input variables (keeping aside the output variables for the moment). Consider the dual cost function of the compound feed industry aiming at depiction of the demand for feed components without taking into account non-component input prices as an example:[2]

The index set $I = \{1,...,n\}$ of the input price vector $\boldsymbol{w} \in \Re^n$ is partitioned into mutually exclusive and exhaustive sectors I^r of length n^r with $r = 1,...,Q$. With each I^r there corresponds a sub-vector of $\boldsymbol{w}$, namely $\boldsymbol{w}^r = (w_i)_{i \in I^r}$. Now let $C(\boldsymbol{w}^1,\boldsymbol{w}^2,\boldsymbol{y}) : \Re^{n+m} \to \Re$ denote

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[*] The same is valid for supply models where the output price or quantity vector is partitioned into no more than two sectors and one of them is ignored in the analysis.

a cost function modelling the economic behavior of the compound feed firm where $\boldsymbol{w}^1$ is the vector of component prices, e. g. the prices of wheat, barley, soy meal, manioc, et c., $\boldsymbol{w}^2$ is the vector of prices of all other inputs like energy, labour, et c., and $\boldsymbol{y}$ is the unpartitioned vector of m output quantities, i. e. of poultry feed, cattle feed, and the like, corresponding with the index set J. Suppose we are only interested in component demand, and data availability precludes the evaluation of $\boldsymbol{w}^2$. In this case, a successful analysis requires component demands $\boldsymbol{x}^1$ derived according to Shephard's lemma to be independent of $\boldsymbol{w}^2$, i. e.

$$\nabla_{w^1} C\left(\boldsymbol{y}, \boldsymbol{w}^1, \boldsymbol{w}^2\right) = \boldsymbol{x}^1\left(\boldsymbol{y}, \boldsymbol{w}^1\right).$$

Assuring this in turn requires the cost function to be of the literally additive form

$$C(\boldsymbol{w}^1, \boldsymbol{w}^2, \boldsymbol{y}) = f(\boldsymbol{w}^1, \boldsymbol{y}) + g(\boldsymbol{w}^2, \boldsymbol{y})$$

or, to proceed step by step, the CES form

$$C\left(\boldsymbol{w}^1, \boldsymbol{w}^2, \boldsymbol{y}\right) = f\left(\boldsymbol{y}\right)\left[g\left(\boldsymbol{w}^1, \boldsymbol{y}\right)^{\rho(y)} + h\left(\boldsymbol{w}^2, \boldsymbol{y}\right)^{\rho(y)}\right]^{\frac{1}{\rho(y)}} \qquad 0 \neq \rho \leq 1,$$

from where it could be argued economically why this model should simplify to the literally additive form above by making substantial substitutability postulates, i. e. by claiming that components and non-components are perfect complements, which is the same as to state that g and h interact linearly and thus $\rho(\boldsymbol{y})$ equals one.

The possibility of economic justification of such a restrictive structural assumption like the CES specification is provided by the concept of separability: based on pioneering work by Goldman and Uzawa,[3] in their extensive monograph *Duality, Separability, and Functional Structure*[4] Blackorby, Primont, and Russell show how an additive structure of many economic functions up to a CES specification can be followed from a separability assumption like e. g. strong input price separability of the dual cost function

$$\frac{\partial}{\partial w_k} \frac{\partial C(\boldsymbol{w}, \boldsymbol{y}) / \partial w_i}{\partial C(\boldsymbol{w}, \boldsymbol{y}) / \partial w_j} = 0 \qquad \forall i \in I^r, j \in I^s, k \notin I^r \cup I^s$$

which is necessary and sufficient for a CES form cost function. By Shephard's lemma, this vivid, easyly applicable assumption means that the ratio between the demand for x_i and x_j stays unaffected by changes in the price of good k. But there is a snag: the proof of the theorem for strong separability equalling a CES specification requires $Q \geq 3$.[5] In other words:

according to Blackorby, Primont, and Russell, the CES specification of the dual cost function presented above cannot be justified by separability.*

To overcome this problem, they establish another condition called ”independence“ that is equivalent with an additive form of the basic function in the binary partition:

$$\frac{\partial}{\partial w_i}\left(\ln \frac{\partial C / \partial w_k}{\partial C / \partial w_j}\right) = \frac{\partial}{\partial w_i}\left(\ln \frac{\partial C / \partial w_l}{\partial C / \partial w_j}\right) = \psi^{ji}\big(C(\boldsymbol{w}, \boldsymbol{y})\big)$$
$$\forall \quad i, j \in I^r \times I^r; \quad k, l \in I^s \times I^s$$

with an arbitrary $\psi^{ji} : \Re^{n+m} \rightarrow \Re$.[6] If one looks at this condition with the eye of an applied economist, it rapidly becomes clear that there is no gain connected with its consideration: in contrast to separability, it is no less abstract and technical than the CES functional structure. One could as well argue about the phenomenal adequacy of the functional structure itself and leave the independence condition aside. It must be concluded that a binary CES cost structure as in the compound feed industry example stated above and – implicitly or explicitly – presupposed in many applied studies remains purely technical and is a priori economically injustifiable.

The goal of this paper is to show that in the binary partition additive functional structure up to a CES specification is well justifiable by separability assumptions for the dual profit function – and only for the dual profit function. This is possible because a well-behaved profit function, other than any other primal or dual economic function, is homogeneous of degree one in both the input and output variables.[7] The presented approach is based on a structural theorem for the profit function closely paralleling a general theorem given by Goldman and Uzawa,[8] that makes it possible to apply a theorem by Blackorby, Primont, and Russell[9] which establishes equality of strong separability and a CES structure on the profit function. First, a preliminary lemma is shown and a condition for the well-behaved profit function is derived from its known properties, which allows generalization of Goldman’s and Uzawa’s structural

* The question of wether the non-component sector could be partitioned into more than one sector is irrelevant here because there is no economic reason for an additional separability assumption inside the non-component sector; Ockham’s razor dictates the choice of the most simple a priori reasonable hypothesis. Anyway, a partition of the non-component sector is impossible because some exogeneous variables like wage influence decisions inside any possible non-component subsector like transportation, stocks, accounting, marketing, the actual mixing process and so on such that mutually exclusive sectors as necessary condition for a partition cannot be established.

theorem for the profit function. Then, this structural theorem is put in concrete terms using Blackorby's, Primont's, and Russell's method.

2 Separability and Functional Structure

After preparing the ground by a lemma that, under certain conditions, proves the existence of a function F that transforms an arbitrary twice continuously differentiable function in another given function, the general theorem by Goldman and Uzawa on the equivalence of a strong separability relation between three arbitrary sections of the exogeneous variables and a CES form of the underlying function is presented, and its proof is repeated in a more extensive way than in the original paper.

2.1 Preliminary Lemmas

LEMMA 1:[10] *Let f,* $g:\Re^n \to \Re$ *be two twice continously differentiable functions of n variables. If the sets* $\{\boldsymbol{x} \in \Re^n | g(\boldsymbol{x}) = \alpha\}$ *are connected by continuously differentiable paths for any* $\alpha > 0$ *and if there exists a function* $\lambda : \Re^n \to \Re$ *such that*

$$f_{x_i} = \lambda \cdot g_{x_i} \qquad \forall i = 1,\ldots,n,$$

then f can be obtained from g by applying a transformation function $F : \Re \to \Re$ *such that*

$$f(\boldsymbol{x}) = F(g(\boldsymbol{x})) \qquad \forall \boldsymbol{x} \in \Re^n.$$

PROOF: To prove the lemma, it suffices to show that, for any $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Re^n$, $g(\boldsymbol{x}_0) = g(\boldsymbol{x}_1)$ implies that $f(\boldsymbol{x}_0) = f(\boldsymbol{x}_1)$ and thus F can be defined as

$$F(\alpha) = \begin{cases} f(\boldsymbol{x}_\alpha) & \text{if } \alpha = g(\boldsymbol{x}_\alpha) \\ 0 & \text{if } \alpha \notin g(\Re^n) \end{cases}$$

because in this case F is well-defined, i. e. if we have $g(\boldsymbol{x}_\beta) = \alpha$ we can as well define $F(\alpha)$ as $g(\boldsymbol{x}_\beta)$.

Let $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Re^n$ with $g(\boldsymbol{x}_0) = g(\boldsymbol{x}_1)$ be given. Then, from our assumption, we find a continuously differentiable path $\varphi : [0,1] \to \Re^n$ in $\{\boldsymbol{x} \in \Re^n | g(\boldsymbol{x}) = \alpha\}$ with $\varphi(0) = \boldsymbol{x}_0, \varphi(1) = \boldsymbol{x}_1$. Applying the chain rule, from g being constant on the path φ it follows that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(\varphi(t)) = \sum_{i=1}^{n} \left(\left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_i(t) \right) g_{x_i}(\varphi(\boldsymbol{x})) \right).$$

Under this condition we obtain for f along φ that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi(t)) = \sum_{i=1}^{n} \left(\left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_i(t) \right) f_{x_i}(\varphi(\boldsymbol{x})) \right) = \lambda(x) \sum_{i=1}^{n} \left(\left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_i(t) \right) g_{x_i}(\varphi(\boldsymbol{x})) \right) = 0.$$

Showing that f is constant along φ, it follows that $f(\boldsymbol{x}_0) = f(\boldsymbol{x}_1)$, which completes the proof. ◊

To apply this lemma on the profit function requires the iso-profit hyperplanes to be connected by continuously differentiable paths in $\Re^n$, which is expressed by the following condition:

CONDITION 2: *Let* $g(\boldsymbol{x}) : \Omega \subset \Re^n_{\geq 0} \to \Re_{\geq 0}$ *be a twice continously differentiable and linearly homogeneous function of n variables. Furthermore, let g be mixed monotonic, i. e., after suitably arranging the indices, say increasing in* $x_1, \ldots, x_{n'}$ *and decreasing in* $x_{n'+1}, \ldots, x_n$ *for* $1 \leq n' < n$. *For* $z_s, z_e \in \Omega$, *the complete hyper-rectangular solid* $\left\{ z \middle| z_{s_i} \leq z_i \leq z_{e_i} \vee z_{e_i} \leq z_i \leq z_{s_i} \quad \forall i = 1, \ldots, n \right\}$ *with corner points* z_s, z_e *lies in* Ω. *Furthermore, for* $z \in \Omega, \lambda > 0$, *let* $\lambda z \in \Omega$ *too, i. e.* Ω *is star-shaped with center* 0. *Then, for any* $z_1, z_2 \in \Omega$ *such that* $g(z_s) = g(z_e) = \alpha \geq 0$ *there existsts a continuously differentiable path* $\lambda : [0,1] \to \Omega$ *with* $\lambda[0,1] \subset g^{-1}(\alpha)$.

PROOF: It suffices to find a path on which g does not vanish, since for a path $\gamma : [0,1] \to \Omega$ with $g(\gamma(t)) > 0, t \in [0,1]$ we know (because of the compactness of the unit interval in conjunction with monotonicity) that $g(\gamma(t))$ is bounded from below by an $\varepsilon > 0$. Consequently, because of the star-shape of Ω, $\tilde{\gamma}(t) : t \mapsto \alpha\gamma(t) / g(\gamma(t))$ defines a differentiable path in Ω too. If z_s and z_e are starting point and end point of γ, respectively, they are also starting and end point of $\tilde{\gamma}$. Homogeneity of g yields that g is constant on $\tilde{\gamma}$ and assumes the value α. Now it only remains to find a path from z_s to z_e where g does not vanish. One finds such a path by decomposing the difference $v = z_e - z_s$ into $v = v_- + v_+$ such that $g(z_s + tv_+)$ is strictly increasing in t and $g(z_s + v_+ + tv_-)$ is strictly decreasing, which is possible with regard to the mixed monotonicity of g. On the compositum of the paths above $z_s \to z_s + v_+ \to z_e = z_s + v_+ + v_-$ the value of g is always greater than α because it at first increases and then decreases to the value α. This path would have to be smoothened at the points where the partial paths meet. That is possible (but tedious and therefore left aside) since g is continuous and hence does not become zero in a neighbourhood of $z_s + v_+$ without leaving the rectangular solid spanned by z_s, z_e, and thus without leaving Ω. ◊

2.2 A General Theorem on Strong Separability and Functional Structure

We adopt the notation of the original article to ease comparism. It has to be mentioned that the formulation in terms of utility does not restrict the generality of the theorem: no specific property of the utility function enters the proof.

THEOREM 3:[11] *Let* $u : \Omega \subset \Re^n \to \Re$ *denote a three times continuously differentiable function with variables* $x_1,\dots,x_n$ *which are partitioned into* Q *sectors* $I^1,\dots,I^Q$ *with* $Q \geq 3$. $u(\boldsymbol{x}) = u(\boldsymbol{x}^1,\dots,\boldsymbol{x}^Q)$ *with* $\boldsymbol{x}^r = (x_i)_{i \in I^r}$. *Then* u *is strongly separable with respect to this partition if and only if it can be written as*

$$u(\boldsymbol{x}) = F\left(u^1(\boldsymbol{x}^1) + \dots + u^Q(\boldsymbol{x}^Q)\right)$$

where F is continuously differentiable and increasing.

PROOF: Necessity immediately follows from the definition of separability. For sufficiency, consider three distinct sectors I^r, I^s, I^t and indices $i \in I^r, j \in I^s, k \in I^t$. Calculation of the partial derivatives in the separability definition*

$$\frac{\partial}{\partial x_k} \frac{\partial u(\boldsymbol{x}) / \partial x_i}{\partial u(\boldsymbol{x}) / \partial x_j} = \left(\frac{u_i}{u_j} \right)_k = 0$$

yields

$$\frac{u_{ik} u_j - u_i u_{jk}}{u_j^2} = 0,$$

from where multiplication with $u_{ik} / u_i u_k$ leads to

$$\alpha \overset{\text{def.}}{=} \frac{u_{ik}}{u_i u_k} = \frac{u_{jk}}{u_j u_k}.$$

* This definition presupposes that the partial derivatives of u do not vanish on Ω.

From this we get independence of $\alpha(x)$ from the choice of indices i, j, k because, according to Young's Theorem, the mixed second order derivatives are symmetric. Note that the equations above essentially require three sectors.

Thus, for i, j, k from distinct sectors, we have

$$u_{ij}(\boldsymbol{x}) = \alpha(\boldsymbol{x}) \cdot u_i(\boldsymbol{x}) \cdot u_j(\boldsymbol{x})$$

and in particular

$$u_{ik}(\boldsymbol{x}) = \alpha(\boldsymbol{x}) \cdot u_i(\boldsymbol{x}) \cdot u_k(\boldsymbol{x}), \tag{1}$$

which, derived with respect to x_l, $l \in I^t$, yields

$$\begin{aligned} u_{ikl} &= \alpha_l \cdot u_i \cdot u_k + \alpha \cdot u_{il} \cdot u_k + \alpha \cdot u_i \cdot u_{kl} \\ &= \alpha_l \cdot u_i \cdot u_k + \alpha \cdot (\alpha \cdot u_i \cdot u_l) \cdot u_k + \alpha \cdot u_i \cdot (\alpha \cdot u_k \cdot u_l) \\ &= \alpha_l \cdot u_i \cdot u_k + 2 \cdot \alpha^2 \cdot u_i \cdot u_k \cdot u_l . \end{aligned}$$

Solved for α_l and divided by u_l we obtain

$$\lambda(\boldsymbol{x}) \overset{\text{def.}}{=} \frac{\alpha_l(\boldsymbol{x})}{u_l(\boldsymbol{x})} = \frac{u_{ikl}(\boldsymbol{x})}{u_i(\boldsymbol{x}) u_k(\boldsymbol{x}) u_l(\boldsymbol{x})} - 2\alpha^2(\boldsymbol{x}).$$

The latter expression is again symmetric in i, k, l, and therefore $\lambda(\boldsymbol{x})$ is independent of the choice of l even if l belongs to another sector than I^t: Let l' be an arbitrary valid index with e. g. $l' \notin I^r \cup I^s$ (which can always be obtained simply by interchanging the sector numbers). Then it follows that

$$\frac{\alpha_{l'}(\boldsymbol{x})}{u_{l'}(\boldsymbol{x})} = \frac{u_{ikl'}(\boldsymbol{x})}{u_i(\boldsymbol{x}) u_k(\boldsymbol{x}) u_{l'}(\boldsymbol{x})} - 2\alpha^2(\boldsymbol{x}) = \frac{u_{il'k}(\boldsymbol{x})}{u_i(\boldsymbol{x}) u_{l'}(\boldsymbol{x}) u_k(\boldsymbol{x})} - 2\alpha^2(\boldsymbol{x}) = \frac{\alpha_k(\boldsymbol{x})}{u_k(\boldsymbol{x})} = \frac{\alpha_l(\boldsymbol{x})}{u_l(\boldsymbol{x})}.$$

Hence, we have the preliminaries to apply Lemma 1.1: $\alpha_k(\boldsymbol{x}) = \lambda(\boldsymbol{x}) u_k(\boldsymbol{x}) \ \forall k$. Thus, there exists a function $\beta : \Re \to \Re$ such that $u(\boldsymbol{x})$ is transformed to $\alpha(\boldsymbol{x})$:

$$\alpha(\boldsymbol{x}) = \beta(u(\boldsymbol{x})).$$

Now we define

$$\tilde{F}(u) \overset{\text{def.}}{=} \int_{u_0}^{u} e^{-\int_{v_0}^{v} \beta(u) \mathrm{d} v} \mathrm{d} u,$$

where $u_0, v_0 \in \Re_{\geq 0}$. This yields

$$\tilde{F}''(u) = \left(e^{-\int_{v_0}^{v} \beta(u)\,\mathrm{d}v} \right)' = -\beta(u) e^{-\int_{v_0}^{v} \beta(u)\,\mathrm{d}v} = -\beta(u)\tilde{F}'(u).$$

It follows that

$$\tilde{F}''(u) + \beta(u)\tilde{F}'(u) = 0. \tag{2}$$

To simplify notation, we write $v(\boldsymbol{x}) = \tilde{F}'(u(\boldsymbol{x}))$. Then, for valid indices i, j from distinct sectors, we obtain $v_i(\boldsymbol{x}) = u_i(\boldsymbol{x})\tilde{F}'(u(\boldsymbol{x}))$ for the partial derivatives and

$$\begin{aligned} v_{ij}(\boldsymbol{x}) &= u_{ij}(\boldsymbol{x})\tilde{F}'(u(\boldsymbol{x})) + u_i(\boldsymbol{x})u_j(\boldsymbol{x})\tilde{F}''(u(\boldsymbol{x})) \\ &= \alpha(\boldsymbol{x})u_i(\boldsymbol{x})u_j(\boldsymbol{x})\tilde{F}'(u(\boldsymbol{x})) + u_i(\boldsymbol{x})u_j(\boldsymbol{x})\tilde{F}''(u(\boldsymbol{x})) && \text{by (1)} \\ &= u_i(\boldsymbol{x})u_j(\boldsymbol{x})\left[\beta(u(\boldsymbol{x}))\tilde{F}'(u(\boldsymbol{x})) + \tilde{F}''(u(\boldsymbol{x}))\right] && \text{by } \alpha(\boldsymbol{x}) = \beta(u(\boldsymbol{x})) \\ &= 0 && \text{by (2).} \end{aligned}$$

Consequently, $v_i(\boldsymbol{x})$ is independent of all variables x_j belonging to other sectors than that which x_i belongs to. Thus, we have e. g. for all $i \in I^r = \{i_1^r, \ldots, i_{n^r}^r\}$

$$\begin{aligned} v_r(\boldsymbol{x}) &= v_r\left(x_1^1, \ldots, x_{n^1}^1, \ldots, x_1^r, \ldots, x_{n^r}^r, \ldots, x_1^Q, \ldots, x_{n^Q}^Q\right) \\ &= v_r\left(\bar{x}_1^1, \ldots, \bar{x}_{n^1}^1, \ldots, x_1^r, \ldots, x_{n^r}^r, \ldots, \bar{x}_1^Q, \ldots, \bar{x}_{n^Q}^Q\right), \end{aligned}$$

where $\bar{\boldsymbol{x}} = \left(\bar{x}_1^1, \ldots, \bar{x}_{n^1}^1, \ldots, \bar{x}_1^r, \ldots, \bar{x}_{n^r}^r, \ldots, \bar{x}_1^Q, \ldots, \bar{x}_{n^Q}^Q\right)$ is constant. Now, $v(\boldsymbol{x})$ can be expressed by successive integration:

$$v(\boldsymbol{x}) - v(\bar{\boldsymbol{x}}) = \sum_{i=1}^{n} \int_{\bar{x}_i}^{x_i} v_i(x_1, \ldots, x_{i-1}, t, \bar{x}_{i+1}, \ldots, \bar{x}_n)\,\mathrm{d}t = \sum_{r=1}^{Q} \overbrace{\sum_{i \in I^r} \int_{\bar{x}_i}^{x_i} v_i(x_1, \ldots, x_{i-1}, t, \bar{x}_{i+1}, \ldots, \bar{x}_n)\,\mathrm{d}t}^{\hat{u}^r(\boldsymbol{x}^r)}.$$

Here, the $\hat{u}^r(\boldsymbol{x})$ depend only on the respective $\boldsymbol{x}^r$. Now $v(\boldsymbol{x})$ equals $\tilde{F}(u(\boldsymbol{x}))$, and with $u^r(\boldsymbol{x}) \overset{\text{def.}}{=} \hat{u}^r(\boldsymbol{x}) + \frac{1}{Q} \cdot v(\bar{\boldsymbol{x}})$ we have found a formulation

$$\tilde{F}(u(\boldsymbol{x})) = \sum_{r=1}^{Q} u^r(\boldsymbol{x}^r),$$

and, since $\tilde{F}$ as an integral over a positive integrand is strictly increasing and hence invertible, we can define $F \overset{\text{def.}}{=} \tilde{F}^{-1}$, and then we obtain from $v(\boldsymbol{x}) = \tilde{F}(u(\boldsymbol{x}))$

$$u(\boldsymbol{x}) = F(v(\boldsymbol{x})) = F\left(\sum_{r=1}^{Q} u^r(\boldsymbol{x}^r)\right). \Diamond$$

3 Strong Separability and Functional Structure of the Dual Profit Function

In this section, Theorem 3 is applied to the dual profit function, and then a more specific structure for the profit function is found. Analogous to the proceeding of Blackorby, Primont, and Russell who perform the same for the dual cost function, three strongly separable price sectors are assumed.[12] The dual profit function is homogeneous of degree one in all variables while the dual cost function is homogeneous of degree one in all prices. The proof of the theorem given by Blackorby, Primont, and Russell heavily relies on the homogeneity property, and the same is valid for the dual profit function analog below. But, in contrast to the cost function, where there is an output quantity sector in addition that is not homogeneous (at least not necessarily), the profit function is homogeneous of degree one in *all* exogeneous variables. Consequently, it is possible to make no distinction between input and output variables. To take account of this generality, we introduce a notation that treats input and output prices symmetrically: $(\boldsymbol{p},\boldsymbol{w}) \equiv (\boldsymbol{z})$ with $\boldsymbol{p} \in \Re^m$, $\boldsymbol{w} \in \Re^n$ and $\boldsymbol{z} \in \Re^{m+n}$. Regarding the fact that the different signs of the strictly increasing or strictly decreasing variables are irrelevant for the proof, it is operational to arrange the index ordering according to the undertaken partition such that the sectors are constituted by compact subsets of *I*.

The theorem for the strongly separable dual profit function has an interesting special case: the case with a total of three partitions, where one of the sectors includes all output prices and the two remaining contain input prices only, or vice versa, namely a separable profit function in the binary *and* extended partition. This constitutes the wanted literally additive profit function whose derivatives for any variable are independent of variables included in another variable sector.

THEOREM 4: *Let* $\Pi(\boldsymbol{p},\boldsymbol{w}) = \Pi(\boldsymbol{z}) : \Omega \to \Re$ *with domain* $\Omega \subset \Re^{m+n}_{>0}$ *and price variables* $\boldsymbol{z} \overset{\text{def.}}{=} (\boldsymbol{p},\boldsymbol{w}) \in \Omega$ *denote a three times continuously differentiable dual profit function, whose variables* $z_1,\ldots,z_n$ *are partitioned into* Q *mutually exclusive and exhaustive sectors* $I = I^1 \cup I^2 \cup \ldots \cup I^Q$ *with* $Q \geq 3$*. Let* Ω *and the iso-profit surface of* $\Pi(\alpha)$, $\alpha > \boldsymbol{0}$*, be connected by differentiable paths. Then,* $\Pi(\boldsymbol{z}) = \Pi(\boldsymbol{z}^1,\ldots,\boldsymbol{z}^Q)$ *with* $\boldsymbol{z}^r = \{z_i\}\ \forall i \in I^r$ *is strongly separable with respect to this partition if and only if it is either of the form*

$$\Pi(z) = \left(\Pi^1(z^1)^\rho + \Pi^2(z^2)^\rho + \ldots + \Pi^Q(z^Q)^\rho\right)^{\frac{1}{\rho}}$$

with $\rho \le 1$ *and* $\rho \ne 0$, *or it is of the form*

$$\Pi(z) = \left(\Pi^1(z^1)^{\rho^1} \cdot \Pi^2(z^2)^{\rho^2} \cdot \ldots \cdot \Pi^Q(z^Q)^{\rho^Q}\right)$$

with real valued ρ^i *such that* $\sum_{i=1}^n \rho^i = 1$.

PROOF: Π_i may denote the partial derivative of Π for z_i, and, analogously, Π_i^r may denote the partial derivative of any Π^r for z_i.

We already know from Theorem 3 that Π is of the form

$$\Pi(z) = F\left(\Pi^1(z^1) + \Pi^2(z^2) + \ldots + \Pi^Q(z^Q)\right).$$

Since Π by definition is homogeneous of degree one, it follows that, for any $i \in I^r, j \in I^s$, the quotients $\Pi_i / \Pi_j = \Pi_i^r / \Pi_j^s$ are homogeneous of degree zero. If i and j belong to different sectors, Euler's The orem yields

$$\sum_{k \in I^r} z_k \left(\frac{\Pi_i^r}{\Pi_j^s}\right)_{z_k} + \sum_{k \in I^s} z_k \left(\frac{\Pi_i^r}{\Pi_j^s}\right)_{z_k} = 0$$

because the derivatives of the quotients for z_k belonging to other sectors than r or s vanish. Calculation of the above derivatives and some manipulations yields

$$\frac{1}{\Pi_i^r} \sum_{k \in I^r} z_k \Pi_{ik}^r = \frac{1}{\Pi_j^s} \sum_{k \in I^s} z_k \Pi_{jk}^s.$$

The left hand side only depends on z^r and the right hand side depends only on z^s, and thus both expressions must be equal to a constant ψ such that

$$\sum_{k \in I^r} z_k \Pi_{ik}^r = \psi \cdot \Pi_i^r \qquad \forall i \in I^r.$$

Euler's Theorem, now applied in the inverse direction,[13] yields that Π_i^r is homogeneous of degree ψ. This ψ is achieved independently of the chosen I^r.

$\sum_{i \in I^r} \Pi_i^r \, \mathrm{d}z_i$ is the total differential of $\Pi^r(z) + \Phi^r$ with a constant Φ^r. Since Π_i^r is homogeneous of degree ψ, we get from Lemma 5 that Φ^r can be chosen such that for $\psi \ne -1$

the expression $\Pi^r(z) + \Phi^r$ is homogeneous of degree $\rho \overset{\text{def.}}{=} \psi + 1$, and for $\psi = -1$ we get real constants σ^r with $\Pi^r(\lambda z) + \Phi^r = \sigma^r \ln(\lambda) + \Pi^r(z) + \Phi^r$ with a constant σ^r.*

Let us at first examine the case of $\psi \neq -1$ or $\rho \neq 0$. In this case, we have $\Pi^r(\lambda z^r) + \Phi^r = \lambda^\rho \left(\Pi^r(z^r) + \Phi^r\right)$ for $\lambda \geq 0$. This yields

$$\Pi^r(\lambda z^r) = \lambda^\rho \Pi^r(z^r) + \left(\lambda^\rho - 1\right)\Phi^r .$$

Regarding the functions $\tilde{\Pi}^r(z^r) \overset{\text{def.}}{=} \Pi^r(z^r) + \Phi^r$ that are homogeneous of degree ρ we obtain $\Pi(z) = F\left(\sum_{r=1}^{3} \tilde{\Pi}^r(z^r) - \Phi^r\right)$. Substituting $\tilde{F}(t) \overset{\text{def.}}{=} F\left(t - \sum_{r=1}^{3} \Phi^r\right)$ yields

$$\Pi(z) = \tilde{F}\left(\sum_{r=1}^{3} \tilde{\Pi}^r(z^r)\right).$$

From $\tilde{\Pi}^r$ being homogeneous of degree ρ and Π being homogeneous of degree one by assumption it follows for $\lambda > 0$ that

$$\lambda \tilde{F}\left(\sum_{r=1}^{3} \tilde{\Pi}^r(z^r)\right) = \lambda \Pi(z) = \lambda \Pi(\lambda z) = \tilde{F}\left(\sum_{r=1}^{3} \tilde{\Pi}^r(\lambda z^r)\right) = \tilde{F}\left(\lambda^\rho \sum_{r=1}^{3} \tilde{\Pi}^r(z^r)\right).$$

Hence, $\tilde{F}$ is homogeneous of degree $1 / \rho$, and it follows

$$\Pi(z) = \tilde{F}(1)\left(\lambda^\rho \sum_{r=1}^{3} \tilde{\Pi}^r(z^r)\right)^{\frac{1}{\rho}}.$$

If we now define $\hat{\Pi}^r(z^r) \overset{\text{def.}}{=} \tilde{F}(1)\left(\tilde{\Pi}^r(z^r)\right)^{1/\rho}$, the $\hat{\Pi}^r(z^r)$ are homogeneous of degree one, and we have

$$\Pi(z) = \left(\lambda^\rho \sum_{r=1}^{3} \hat{\Pi}^r(z^r)\right)^{\frac{1}{\rho}},$$

and hence the theorem is shown for the case that $\psi \neq -1$.

Now, if $\psi = -1$ we obtain $\Pi^r(\lambda z) + \Phi^r = \ln(\lambda)\sigma^r + \Pi^r(\lambda z) + \Phi^r$, as stated above. Applying the e-function yields

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* At this point, Blackorby, Primont, and Russell do not respect the case $\psi = -1$; later they try to overcome this by establishing it as a limiting case in a dubious way.

$$e^{\Pi^r(\lambda z^r)} = \lambda^{\sigma^r} e^{\Pi^r(z^r)}.$$

Thus, $\overline{\overline{\Pi}}^r(z^r) \overset{\text{def.}}{=} e^{\Pi^r(z^r)}$ is homogeneous of degree σ^r, and from $\Pi^r(z^r) = \ln \overline{\overline{\Pi}}^r$ for all $r = 1,\dots,Q$ follows

$$\Pi(z) = F\left(\sum_{r=1}^{Q} \ln \overline{\overline{\Pi}}^r(z^r)\right) = F^*\left(\prod_{r=1}^{Q} \overline{\Pi}^r(z^r)^{\sigma^r}\right)$$

with $F^*(t) \overset{\text{def.}}{=} F(\ln(t))$, and the $\overline{\Pi}^r(z^r) \overset{\text{def.}}{=} \left(\overline{\overline{\Pi}}^r(z^r)\right)^{1/\sigma^r}$ being positively linearly homogeneous for all $r = 1,\dots,Q$. Since Π is linearly homogeneous, we have

$$\lambda F^*\left(\prod_{r=1}^{Q} \overline{\Pi}^r(z^r)^{\sigma^r}\right) = F^*\left(\lambda^{\left(\sum_{r=1}^{Q}\sigma^r\right)} \prod_{r=1}^{Q} \overline{\Pi}^r(z^r)^{\sigma^r}\right).$$

Consequently, F^* is homogeneous of degree $\left(\sum_{r=1}^{Q}\sigma^r\right)^{-1}$. It follows

$$\Pi(z) = F^*(1)\prod_{r=1}^{Q} \overline{\Pi}^r(z^r)^{\sigma^r} \qquad \text{with } \rho^i = \sigma^i\left(\sum_{r=1}^{Q}\sigma^r\right)^{-1}. \;\Diamond$$

4 Appendix

The inverses of some general properties of homogeneous functions commonly known only in one direction are added: the fact that partial derivatives of homogeneous functions are homogeneous for one degree fewer than the original function and the inverse of Euler's Theorem.

LEMMA 5: *Let $f(\boldsymbol{x}) : \Omega \subset \Re^n \to \Re$ denote a twice continuously differentiable function, defined on a connected open subset $\Omega \subset \Re^n$. If all partial derivatives $f_{x_i} \equiv \partial f(\boldsymbol{x})/\partial x_i$ are homogeneous of degree η for all $i = 1,\dots,n$, then there exists a constant $K \in \Re$ such that, for $\eta \neq 0$, $f + K$ is homogeneous of degree $\eta + 1$, and, for $\eta = -1$, the function f satisfies $f(\lambda \boldsymbol{x}) = \ln(\lambda)K + f(\boldsymbol{x})$ for any $\lambda > 0$ where K is a constant that is independent from $\boldsymbol{x}$.*

PROOF: Let us examine the function $g(\lambda) \overset{\text{def.}}{=} f(\lambda \boldsymbol{x})$. The chain rule yields

$$g'(\lambda) = \sum_{i=1}^{n} x_i f_{x_i}(\lambda \boldsymbol{x})$$

and we see that g‘ is homogeneous of degree η by assumption.

We have $g'(\lambda) = \lambda^{\eta} g'(1)$, which yields $g(\lambda) = \lambda^{\eta+1} g'(1) + K$. From this equation we deduce for $\eta \neq -1$ that

$$f(\lambda \boldsymbol{x}) = g(\lambda) = \lambda^{\eta+1} \frac{g'(1)}{\eta+1} + K \tag{3}$$

and, for $\eta = -1$,

$$f(\lambda \boldsymbol{x}) = g(\lambda) = K \cdot \ln \lambda + C. \tag{4}$$

Substituting $\lambda = 1$, we see that $C = f(\boldsymbol{x})$, and substituting $\lambda = e$ reveals that $K = f(e\boldsymbol{x}) - f(\boldsymbol{x})$ for some $\boldsymbol{x}$ in equation (4). From the homogeneity of the partial deravatives of f we get $\left(f(e\boldsymbol{x}) - f(\boldsymbol{x})\right)_{x_i} = ef_{x_i}(e\boldsymbol{x}) - f_{x_i}(\boldsymbol{x}) = 0$, and we see that K does not depend on $\boldsymbol{x}$.

In equation (3) we get from observing the case $\lambda = 1$

$$\begin{aligned}
\frac{\partial K}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(f(\boldsymbol{x}) - \frac{g'(1)}{\eta+1} \right) \\
&= f_{x_j} - \frac{1}{\eta+1} \sum_{i=1}^{n} \frac{\partial}{\partial x_j}\left(\boldsymbol{x}_i f_{\boldsymbol{x}_i}(\boldsymbol{x}) \right) \\
&= f_{x_j} - \frac{1}{\eta+1}\left(\left(\sum_{i=1}^{n} \boldsymbol{x}_i f_{\boldsymbol{x}_i \boldsymbol{x}_j}(\boldsymbol{x}) \right) + f_{\boldsymbol{x}_j}(\boldsymbol{x}) \right) \\
&= f_{x_j} - \frac{1}{\eta+1}\left(\left(\sum_{i=1}^{n} \boldsymbol{x}_i f_{\boldsymbol{x}_j \boldsymbol{x}_i}(\boldsymbol{x}) \right) + f_{\boldsymbol{x}_j}(\boldsymbol{x}) \right)
\end{aligned}$$

and, applying Euler’s Theorem on $f_{\boldsymbol{x}_j}$,

$$= f_{x_j} - \frac{1}{\eta+1}\left(\eta \cdot f_{\boldsymbol{x}_j}(\boldsymbol{x}) + f_{\boldsymbol{x}_j}(\boldsymbol{x}) \right) = 0.$$

Hence, K in equation (3) does not depend on $\boldsymbol{x}$ either. ◊

LEMMA 6 (INVERSE OF EULER’S THEOREM)**:** *Given a continuously differentiable function* $f : \mathfrak{R}^n \to \mathfrak{R}$. *Let there be an* $\eta \in \mathfrak{R}$ *where*

$$\sum_{i=1}^{n} x_i \frac{\partial f(\boldsymbol{x})}{\partial x_i} = \eta \cdot f(\boldsymbol{x}).$$

Then, f is homogeneous of degree η.

PROOF: Given $\boldsymbol{x} \in \Re^n$. For any $\lambda > 0$, it immediately follows from the presupposed equation that $f(\lambda \mathbf{0}) = \lambda^\eta f(\mathbf{0}) = 0$, and thus it may be assumed that $\boldsymbol{x} \neq \mathbf{0}$.

Let us define $g(t) \overset{\text{def.}}{=} f(t\boldsymbol{x})$ for any $t > 0$. Then, the chain rule yields

$$g'(t) = \sum_{i=1}^{n} x_i \frac{\partial f(\boldsymbol{x})}{\partial x_i}(tx_i) = \frac{1}{t}\sum_{i=1}^{n}(tx_i)\frac{\partial f(\boldsymbol{x})}{\partial x_i}(tx_i) = \frac{\eta}{t} f(t\boldsymbol{x}) = \frac{\eta}{t} g(t).$$

Consequently, g satisfies the differential equation $g'(t)/g(t) = \eta/t$. For $\eta = 0$, g must be constant and we obtain $g(t) = g(1) = f(\boldsymbol{x})$. For $\eta \neq 0$ we have the general solution $g(t) = Ct^\eta$; thus, from $g(1) = f(\boldsymbol{x})$ it follows that $g(t) = t^\eta f(\boldsymbol{x})$. In both cases it holds that $f(t\boldsymbol{x}) = t^\eta f(\boldsymbol{x})$. ◊

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[1] Contact: mail@fritzfeger.de

[2] See among numerous others e. g. PEETERS/SURRY 1993#.

[3] S. M. GOLDMAN / H. UZAWA 1964: *A Note on Separability in Demand Analysis*. Econometrica, Vol. 32, No. 3, pp. 387-398.

[4] Charles BLACKORBY / Daniel PRIMONT / Robert RUSSELL 1978: *Duality, Separability, and Functional Structure*: *Theory and Economic Applications*. North Holland; New York, Oxford, Shannon.

[5] See section 2.2.

[6] See BLACKORBY/PRIMONT/RUSSELL 1978: 161-165, where the independence condition is formulated for the utility/production function.

[7] See proof of Theorem 4 below.

[8] See GOLDMAN/UZAWA 1964.

[9] BPR#

[10] GU#

[11] GU#

[12] BPR#

[13] See appendix#